CHARACTERIZATION OF 1-QUASI-GREEDY BASES

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ABSTRACT. We show that a (semi-normalized) basis in a Banach space is quasi-greedy with quasi-greedy constant equal to 1 if and only if it is unconditional with suppression-unconditional constant equal to 1.

1. INTRODUCTION AND BACKGROUND

Let $(\mathbb{X}, \|\cdot\|)$ be an infinite-dimensional Banach space, and let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^\infty$ be a semi-normalized basis for $\mathbb{X}$ with biorthogonal functionals $(\mathbf{e}_n^*)_{n=1}^\infty$. The basis $\mathscr{B}$ is *quasi-greedy* if for any $x \in \mathbb{X}$ the corresponding series expansion,

$$x = \sum_{n=1}^\infty \mathbf{e}_n^*(x)\mathbf{e}_n$$

converges in norm after reordering it so that the sequence $(|\mathbf{e}_n^*(x)|)_{n=1}^\infty$ is decreasing. Wojtaszczyk showed [6] that a basis $(\mathbf{e}_n)_{n=1}^\infty$ of $\mathbb{X}$ is quasi-greedy if and only if the greedy operators $\mathscr{G}_N \colon \mathbb{X} \to \mathbb{X}$ defined by

$$x = \sum_{j=1}^\infty \mathbf{e}_j^*(x)\mathbf{e}_j \mapsto \mathscr{G}_N(x) = \sum_{j\in\Lambda_N(x)} \mathbf{e}_j^*(x)\mathbf{e}_j,$$

where $\Lambda_N(x)$ is any N-element set of indices such that

$$\min\{|\mathbf{e}_j^*(x)| \colon j \in \Lambda_N(x)\} \geq \max\{|\mathbf{e}_j^*(x)| \colon j \notin \Lambda_N(x)\},$$

are uniformly bounded, i.e.,

$$\|\mathscr{G}_N(x)\| \leq C\|x\|, \quad x \in \mathbb{X}, N \in \mathbb{N}. \tag{1.1}$$

for some constant C independent of x and N. Note that the operators $(\mathscr{G}_N)_{N=1}^\infty$ are neither linear nor continuous, so this is not just the Uniform Boudedness Principle!

Obviously, If (1.1) holds then there is a (possibly different) constant $\tilde{C}$ such that

$$\|x - \mathscr{G}_N(x)\| \leq \tilde{C}\|x\|, \quad x \in \mathbb{X}, N \in \mathbb{N}. \tag{1.2}$$

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We will denote by C_w the smallest constant such that (1.1) holds, and by C_t the least constant in (1.2). It is rather common (cf. [2,4]) and convenient to define the *quasi-greedy constant* C_{qg} of the basis as

$$C_{qg} = \max\{C_w, C_t\}.$$

If $\mathscr{B}$ is a quasi-greedy basis and C is a constant such that $C_{qg} \le C$ we will say that $\mathscr{B}$ is C*-quasi-greedy*.

Recall also that a basis $(\mathbf{e}_n)_{n=1}^\infty$ in a Banach space $\mathbb{X}$ is *unconditional* if for any $x \in \mathbb{X}$ the series $\sum_{n=1}^\infty \mathbf{e}_n^*(x)\mathbf{e}_n$ converges in norm to x regardless of the order in which we arrange the terms. The property of being unconditional is easily seen to be equivalent to that of being *suppression unconditional*, which means that the natural projections onto any subsequence of the basis

$$P_A(x) = \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n, \quad A \subset \mathbb{N},$$

are uniformly bounded, i.e., there is a constant K such that for all $x = \sum_{n=1}^\infty \mathbf{e}_n^*(x)\mathbf{e}_n$ and all $A \subset \mathbb{N}$,

$$\left\| \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n \right\| \le K \left\| \sum_{n=1}^\infty \mathbf{e}_n^*(x)\mathbf{e}_n \right\|. \tag{1.3}$$

The smallest K in (1.3) is the *suppression unconditional constant* of the basis, and will be denoted by K_{su}. Notice that

$$K_{su} = \sup\{\|P_A\| \colon A \subseteq \mathbb{N} \text{ is finite}\} = \sup\{\|P_A\| \colon A \subseteq \mathbb{N} \text{ is cofinite}\}.$$

If a basis $\mathscr{B}$ is unconditional and K is a constant such that $K_{su} \le K$ we will say that $\mathscr{B}$ is K*-suppression unconditional*.

Quasi-greedy bases are not in general unconditional; in fact, most classical spaces contain conditional quasi-greedy bases. Wojtaszczyk gave in [6] a general construction (improved in [5]) to produce quasi-greedy bases in some Banach spaces. His method yields the existence of conditional quasi-greedy bases in separable Hilbert spaces, in the spaces ℓ_p and $L_p[0,1]$ for $1 < p < \infty$, and in the Hardy space H_1. Dilworth and Mitra showed in [3] that ℓ_1 also has a conditional quasi-greedy basis. In spite of that, quasi-greedy bases preserve some vestiges of unconditionality and, for instance, they are *unconditional for constant coefficients* (see [6]).

Conversely, unconditional bases are always quasi-greedy. To be precise, if $\mathscr{B}$ is K-suppression unconditional then $\mathscr{B}$ is K-quasi-greedy. In particular, unconditional bases with $K_{su} = 1$ are quasi-greedy with $C_w = 1$. Our aim is to show the converse of this statement, thus characterizing 1-quasi-greedy bases. The related problem of characterizing bases that are 1-greedy was solved in [1]. This question is relevant since the optimality in the constants of greedy-like bases seems to improve the properties

of the corresponding basis. Indeed, in the ”isometric case" greedy bases gain in symmetry (they are invariant under greedy permutations instead of merely democratic). Our result reinforces this pattern by showing that ”isometric” quasi-greedy basis are not merely unconditional for constant coefficients but unconditional.

2. The Main Theorem and its Proof

As a by-product of their research on unconditionality-type properties of quasi-greedy bases, Garrigós and Wojtaszczyk [5] have shown that bases in Hilbert spaces with $C_w = 1$ are orthogonal. A direct proof of their result can be obtained as follows.

Let $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ be a basis in a (real or complex) Hilbert space with $C_w = 1$. Then, if $|\varepsilon| = 1$, $0 < t < 1$, and $i \neq j$,

$$\|\mathbf{e}_i\|^2 \leq \|\mathbf{e}_i + \varepsilon t \mathbf{e}_j\|^2 = \|\mathbf{e}_i\|^2 + 2t\Re(\varepsilon\langle \mathbf{e}_i, \mathbf{e}_j\rangle) + t^2\|\mathbf{e}_j\|^2.$$

Simplifying,

$$-2\Re(\varepsilon\langle \mathbf{e}_i, \mathbf{e}_j\rangle) \leq t\|\mathbf{e}_j\|^2.$$

Choosing ε such that $\varepsilon\langle \mathbf{e}_i, \mathbf{e}_j\rangle = -|\langle \mathbf{e}_i, \mathbf{e}_j\rangle|$ and letting t tend to zero we obtain $|\langle \mathbf{e}_i, \mathbf{e}_j\rangle| = 0$.

A strengthening of this argument leads to the following generalization of Garrigós-Wojtaszczyk's result.

Theorem 2.1. *A quasi-greedy basis $(\mathbf{e}_n)_{n=1}^{\infty}$ in a Banach space $\mathbb{X}$ is quasi-greedy with $C_w = 1$ if and only if it is unconditional with suppression unconditional constant $K_{su} = 1$.*

Proof. We need only show that if x and y are vectors finitely supported in $(\mathbf{e}_n)_{n=1}^{\infty}$ with disjoint supports then $\|x\| \leq \|x + y\|$. This readily implies that $(\mathbf{e}_n)_{n=1}^{\infty}$ is unconditional with suppression unconditional constant $K_{su} = 1$.

Suppose that this in not the case and that we can pick x, $y \in \mathbb{X}$ finitely and disjointly supported in $(\mathbf{e}_n)_{n=1}^{\infty}$ with $\|x + y\| < \|x\|$. Consider the function $\varphi\colon \mathbb{R} \to [0, \infty)$ defined by

$$\varphi(t) = \|x + ty\|.$$

Using the definition, it is straightforward to check that φ is a convex function on the entire real line. Moreover, $\varphi(0) = \|x\|$ and, by assumption, $\varphi(1) < \|x\|$. Therefore, $\varphi(t) < \|x\|$ for all $0 < t < 1$. Choosing $t \in (0,1)$ small enough we have $x = \mathscr{G}_N(x + ty)$, where N is the cardinal of the support of x. Consequently, for such a t,

$$\|x + ty\| = \varphi(t) < \|x\| = \|\mathscr{G}_N(x + ty)\| \leq \|x + ty\|,$$

where we used the hypothesis on the quasi-greedy constant of the basis to obtain the last inequality. This absurdity proves the result. □

We close with some consequences of Theorem 2.1, which need no further explanation.

Corollary 2.2. *Suppose $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ is a basis in a Banach space $\mathbb{X}$ with $C_w = 1$. Then $C_t = 1$; in particular $\mathscr{B}$ is 1-quasi-greedy.*

Corollary 2.3. *If a basis $(\mathbf{e}_n)_{n=1}^{\infty}$ in a Banach space $\mathbb{X}$ is 1-quasi-greedy then it is 1-suppression unconditional.*

Corollary 2.4. *Suppose $\mathscr{B} = (\mathbf{e}_n)_{n=1}^{\infty}$ is a basis in a Banach space $(\mathbb{X}, \|\cdot\|)$. Then $\mathbb{X}$ admits an equivalent norm $|||\cdot|||$ so that $\mathscr{B}$ is 1-quasi-greedy in the space $(\mathbb{X}, |||\cdot|||)$ if and only if $\mathscr{B}$ is unconditional.*

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REFERENCES

[1] F. Albiac and P. Wojtaszczyk, *Characterization of 1-greedy bases*, J. Approx. Theory **138** (2006), no. 1, 65–86.
[2] S. J. Dilworth, N. J. Kalton, D. Kutzarova, and V. N. Temlyakov, *The thresholding greedy algorithm, greedy bases, and duality*, Constr. Approx. **19** (2003), no. 4, 575–597.
[3] S. J. Dilworth and D. Mitra, *A conditional quasi-greedy basis of l_1*, Studia Math. **144** (2001), no. 1, 95–100.
[4] G. Garrigós, E. Hernández, and T. Oikhberg, *Lebesgue-type inequalities for quasi-greedy bases*, Constr. Approx. **38** (2013), no. 3, 447–470.
[5] G. Garrigós and P. Wojtaszczyk, *Conditional quasi-greedy bases in Hilbert and Banach spaces*, Indiana Univ. Math. J. **63** (2014), no. 4, 1017–1036.
[6] P. Wojtaszczyk, *Greedy algorithm for general biorthogonal systems*, J. Approx. Theory **107** (2000), no. 2, 293–314.

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